TECHNICAL REVIEW – SIMPLE MATHEMATICAL MODELS WITH VERY COMPLICATED DYNAMICS

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**Topic of Study**

The following is a review of the above titled review article. The author broadly covers the following areas as they relate to simple difference equations which display unexpectedly complex behavior: compilation of non-linear difference equation phenomena, open questions with an intent to spur further investigation, practical applications to all manner of scientific fields, current [for the year 1976] writings on how chaotic behavior can appear in second order or higher difference equations, and requests the material be introduced in academia as a topic for broader mathematical enrichment.

**Introduction**

Looking back on mathematical advances, it seems that the bulk of American cultural focus has been placed on topics that were studied in primary school (likely because it’s the most relatable to those who’ve learned it). In this regard, not much has changed since the days of Newton and Gauss. The most prevalent courses are clearly in the topics of: arithmetic, algebra, geometry, and calculus. Homework and exam questions that are worked out in these truncated, piecemealed examples, encourage students to unintentionally take away two lessons. One, that all problems are strictly deterministic and right answers can be found. Two, directly implementing an idea leads to a nice, clean, solution is the norm to be expected. However, as early as 1976, Robert May’s article, focusing on the early stages of chaos theory, invites his audience to challenge these ideas. His focus on extremely simple equations that he then uses to show wild mathematical behavior, remove the reader from the comfortable armchair built by years of calculation focused mathematical education, and throw them into a world where the alteration of a parameter by values as minute as even 0.00001 can send values from a stable course, onto a tumbling journey to an unpredictable nowhere. Mathematics isn’t as clear cut and predictable as it may have seemed! Math education hasn’t prepared students for this journey.

The relevance of this article lies in three drastically different domains. First, it gives a tourist guide to chaos theory and simple non-linear difference equation trajectories and patterns. Second, it outlines the value of mathematically modeling these equations for practical applications in a wide variety of scientific fields. Three, shows how very little mathematical education prepares one to study this field, while also requesting it be thrust upon students earlier on in their education to open their minds to the complexity of mathematical behavior that is utterly missed by computationally focused curricula – something that still persists today.

**Methodology and Technical Review Aspects**

May begins by introducing the idea of simple first-order difference equation which helps to clearly define the first question a reader (and likely non-mathematicians) may have. Namely, what is a difference equation? This is done by providing explaining equation (1),

where is broadly considered to be the population. He does a good job ensuring that the reader is aware of the primary usage of the variable, but also notes that pigeon holing it to only that usage would diminish its value; effectively, he gives an anchoring point for what he’ll develop into a more difficult idea while still leaving it otherwise modular.

In the next paragraph, he introduces the logistic equation

His introduction is concise and well-articulated. In the equation above, the reader must pause and attempt to understand his asserted statements, but such pondering is required if the reader is to begin to more thoroughly grasp this concept. Understanding why is a requirement, and toying with the numerical necessity of are important exercises in appreciating where the paper is heading. The only statement that needs further elaboration is why it is disadvantageous to require to be within some interval. This could be elaborated on further in the applications section, but isn’t. One is left to trust him on this point, which means his point is not supported.

Under the heading of “Dynamic properties of equation (1)” he introduces equilibria, i.e., fixed points. The algebra is correct and the equation clear, meaning the reader now can easily understand how they’re calculated – setting the equations equal to and solving (essentially, when does the equation equal the value you’ve plugged into it, meaning it will iteratively stay the same?). He also further elaborates this point by introducing the graphical interpretation of this point as values of equation (1) that cross the line [in this case technically ] (again, when is the function equal to its input value). He then introduces the point of stability. Preferably, stability would be defined first, i.e., he would mention that stability means iterations of that value into the difference equation would be attracted or repelled away from fixed points, and then give the numerical interpretation/significance of slopes. He however, does the opposite. This requires the reader to re-read the paragraph in order to piece together the complete meaning of stability as it relates to slope. Also, it should be noted that equation (6) is poorly defined. Combined with figure 1, it is clear that the author means the value of the dashed line at is the slope at the intersection point between the line and the difference equation. But the expression

introduces notation that is not yet meaningful. What is the (1) above lambda? How is the slope of our function with respects to multiplied by (by the way, which ?) meaningful? If the intent was a truncated canonical form, then is clearly missing from the expression as the dashed lines will clearly cross the intercept at a non-zero point. Simply put, equation (6) introduces more confusion than it needs to – it could have been mentioned that the derivative of at an equilibrium point is valuable for calculating its stability and is written as . Otherwise, the many citations near this formula and its explanation, along with the graphical representations help the reader to start to see the correlation between the slope at fixed points and stability. (Again, fixed points are intersections between the curve and the line.)

With equations (7) and (8) the author introduces the idea of iteration of the functions. He explains what is implied by the earlier lambda notation. This however, is clear and his two lines explaining , is a critical point he doesn’t leave to the reader to contemplate further, which is greatly beneficial for the reader’s understanding. Also, his link between the graphical representation of the iterated solutions (figure 2) and the fixed-point equation greatly aid in building a geometric intuition of the concepts of iterated functions and fixed points. Lastly on this point, his mentioning that any fixed point of the initial function, is then a “degenerate” case of the second, i.e., if the point is fixed after one iteration, then it will obviously remain fixed for a second. All of these concepts are introduced and explained well. He makes a remark that doesn’t seem particularly important for the rest of the paper but is interesting nonetheless: .

There is, however, an error in the paper pertaining to the graphs shown in the figures. To achieve : . At an initial value of , we see a graph that is better viewed as Figure 3. Likewise, the graph for Figure 2 is actually the image listed as figure 3. Effectively the images should be swapped. Luckily this doesn’t detract too much from the significance of the findings. Regardless, the graphical intuition is built for the reader.

The author then does a wonderful job of explaining how the geometric interpretation matches with the birth of bifurcation points. He also introduces the bifurcation doubling process mentioning the regions prior to where one only has periodic orbits, and then the transition past that point that sees the infinite possibility of orbits with any number of periodic points. To better understand this idea, one can think of the difference between say a very large repeating string of digits the decimal (the periodic repeating orbits), and an irrational number (i.e., a decimal that goes on forever but never repeats) – but instead with periods. It is unfortunate that at the time this paper was written, that the limits of technology wouldn’t allow for high resolution photos, GIFs, or videos to be attached to research papers. An expansion or live update of this material, as seen in [this](https://www.youtube.com/watch?v=ovJcsL7vyrk) YouTube video (4:00-6:20) done by the user “Veritasium”, beautifully outlines what May is trying to do with three images, table (3) that shows the ever shrinking size of sink windows, and a block of text. All in all, given the technological limits of his time, the method he goes about to present these findings is extremely well done – it allows him to finally introduce one of the main ideas of the paper, period three implies chaos.

The idea he moves to after is how depending on the parameter value of , different portions of the interval will be drawn (mapped) into unique orbits. Therefore, this process, that for higher values of , that appears sporadic and random, is actually well defined and deterministic. Hence, this idea of chaos can be intuitively thought of as something that naturally appears to mimic random (i.e., stochastic) processes. This brings the title of his paper into full view – extremely simple equations (e.g., ) give very complicated dynamics. Also, something that is of extremely interesting note, is that as the value of increases (his Table 3), we see the window (the far-left column in the table) that draws values into a specific orbit, gets smaller and smaller. We also notice, that at specific values of the various parameters, regions switch between producing stable periods, and producing chaotic non-repeating regions. Analytically, this means that any cycle, given the right initial conditions, can be produced with a periodic orbit of . With being any positive integer, and , it is clear that any period of any length can be established so long as just the right minutely precise value for is chosen in the above logistic equation. Lastly, he mentions pitchfork and tangent bifurcations. His methodology isn’t very precise in the paper, but he builds a good graphically intuition for it – namely, when values of are chosen with a certain number of iterations the peaks and valleys of the resulting polynomial graph will either cross or be tangent to the line. When crossed, this is where we more typically observe orbits, and when tangent we see orbits. If this is carried out for more iterations, the curves will repeat themselves and we can then see the combination of .

Of maybe less consequence, but still an intriguing display of mathematical consistency and elegance, the author discusses how one can list out the various periodic cycles as one increases the value for . That’s to say, we can list out the various periodic orbits in order. This might seem obvious, as different periods would come at different points, but given that this is mapping from an uncountable infinity, , to a list of possible orbit sizes in a countable infinity, , one can again appreciate the results of the last paragraph: namely that windows (i.e., intervals ) lead beautifully to some orbital value in . (As an interesting aside, there is a link between and the Fibonacci series!) These can also be precisely calculated to help establish exactly which intervals lead to which periodic cycles. It can also be noted, that because the various iterations and values of produce graphs that give rise to various bifurcation points, and that this process is continuous, that even in the chaotic region of values for , we can still expect periodic cycles to exist therein.

In his section, Mathematical Curiosities, he identifies one of the primary practical use cases for setting . Because it’s known that this causes chaotic behavior, which produces non-repeating patterns, it is an effective random number generator. If one were to try to do the same processes with say , it may be more challenging to get to the th decimal place. However, using simple iterations of a program (like MS Excel), one can easily produce thousands of randomly generated numbers almost instantly between zero and one. To end the section, he introduces another relatively simple equation,

that he then again uses to show can produce chaotic behavior. The details of this are less important than the idea: there are many if not infinite simple equations that one could evaluate that then lead to chaotic behavior! Therefore, one can be naturally led to the thought that perhaps chaos isn’t merely a theoretically interesting point, but instead something that can have profound real-world importance and application. This is precisely the purpose of his final content driven section.

In his, Applications, section, he introduces Lorenz’s butterfly effect hypothesis, and talks about how even a perfectly deterministic model that actually is completely correct in how it models the environment, could be led hopelessly astray by minute errors in the provided initial conditions to the equation. This idea has sense been confirmed. Weather forecasting is a classic example of chaos theory in action – no matter how well the models are built and how much data is collected, the chaotic nature of the weather will inevitably lead to the models not matching how things are panning out in reality.

Lastly, May, ends his article by calling for introducing students to this type of equation and thinking earlier on in mathematics education. His point is well made. Even current mathematics education consists mainly of only algebra, trigonometry, calculus, and discrete mathematics. None of these courses introduce the idea of functions and equations that produce wildly complicated results that can’t be simply mapped to or seen on a graph – it isn’t until a graduate level course in mathematics that one would be exposed to these ideas at all! Hence, his claim is quite insightful, “The mathematical intuition so developed ill equips the student to confront the bizarre behaviour exhibited by the simplest of discrete nonlinear systems, such as equation (3).” (May, 1976).

**Conclusion**

Altogether, Robert May does a good job of introducing the reader to the main topics of complex dynamics of difference equations, bifurcation points, periodic orbits, and chaotic behavior, while also nicely introducing supplementary side examples that begin to scratch the surface of the mathematical complexity that lies under what would otherwise possibly be brushed over as a painfully basic equation not worthy of further investigation. His distillation of a wide range of scientific articles surely helped to bring chaotic behavior to the forefronts of scientists and mathematicians’ minds; thereby, planting interest in and seeding progress in several fields.

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